

Remarks on the statistical origin of the geometrical formulation of quantum mechanics

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Abstract

A quantum system can be entirely described by the Kähler structure of the projective space $\mathbb{P}(\mathcal{H})$ associated to the Hilbert space \mathcal{H} of possible states; this is the so-called geometrical formulation of quantum mechanics.

In this paper, we give an explicit link between the geometrical formulation (of finite dimensional quantum systems) and statistics through the natural geometry of the space \mathcal{P}_n^\times of non-vanishing probabilities $p : E_n \rightarrow \mathbb{R}_+^*$ defined on a finite set $E_n := \{x_1, \dots, x_n\}$. More precisely, we use the Fisher metric g_F and the exponential connection $\nabla^{(1)}$ (both being natural statistical objects living on \mathcal{P}_n^\times) to construct, via the Dombrowski splitting Theorem, a Kähler structure on $T\mathcal{P}_n^\times$ which has the property that it induces the natural Kähler structure of a suitably chosen open dense subset of $\mathbb{P}(\mathbb{C}^n)$.

As a direct physical consequence, a significant part of the quantum mechanical formalism (in finite dimension) is encoded in the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$.

1 Introduction

In quantum mechanics, it is well known that if ψ_1, ψ_2 are two collinear vectors belonging to the Hilbert space \mathcal{H} describing the possible states of a quantum system, then they are physically equivalent. This means that the true configuration space of quantum mechanics is the projective space $\mathbb{P}(\mathcal{H})$, and this leads to a formulation of quantum mechanics uniquely based on the geometry of $\mathbb{P}(\mathcal{H})$ (see [2] for a detailed discussion). In this formulation, the dynamics is governed by the Fubini-Study symplectic form, and the observables are no more operators, rather functions on the projective space having the particularity to preserve, in some sense, the Fubini-Study metric; eigenstates are critical points of the observable functions, and the corresponding critical values are the eigenvalues. In other words, quantum mechanics can be completely formulated in terms of the Kähler structure of the projective space $\mathbb{P}(\mathcal{H})$. Subsequently, we shall refer to this formulation as the *geometrical formulation* of quantum mechanics.

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The geometrical formulation, although complete and very elegant, has a very disappointing feature : it doesn't explain, nor even justify, why its probabilistic interpretation, based on the Fubini-Study metric, is consistent. Consistency is only something that one observes once the computations are done, and, somehow, this appears as "magical". But of course, magic doesn't exist, and this is clearly the sign that there is something in the geometrical formulation that we still don't understand. This concerns the various formulas where probabilities are expected of course, but more generally, it concerns our understanding of the link between the Kähler structure of the projective space $\mathbb{P}(\mathcal{H})$, and statistics.

This link has, until now, attracted very little attention in the existing literature, and its nature is still quite mysterious. The only mention (to our knowledge) of a possible link between the Kähler structure of the projective space, and statistics, comes from *information geometry*¹, where it is argued that the Fubini-Study metric (at least for $\mathbb{P}(\mathbb{C}^n)$), is a kind of "quantum analogue" of the famous *Fisher metric*, the latter being a metric living on the space \mathcal{P}_n^\times of non-vanishing probabilities $p : E_n \rightarrow \mathbb{R}$ defined on a finite set $E_n = \{x_1, \dots, x_n\}$. Such "link" is, however, only an analogy, and in particular, it has no precise mathematical formulation.

Nevertheless, the idea that the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ may originate from the intrinsic geometry of a statistical manifold², is an appealing one, and the main observation of this paper is that it is "almost" true. More precisely, we show that the restriction of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ to the open dense subset³ $\mathbb{P}(\mathbb{C}^n)^\times := \{[z_1, \dots, z_n] \in \mathbb{P}(\mathbb{C}^n) \mid z_k \neq 0 \ \forall k = 1, \dots, n\}$ can be completely recovered from the statistical manifold \mathcal{P}_n^\times , supplemented with the Fisher metric g_F and from another important statistical object living naturally on \mathcal{P}_n^\times , namely the *exponential connection* $\nabla^{(1)}$. Together, these geometrical objects form a triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ which allows the construction of an almost Hermitian structure $(T\mathcal{P}_n^\times, G, J)$ on $T\mathcal{P}_n^\times$; the construction, which is due to Dombrowski (see [3]) is straightforward. For $u_p \in T_p\mathcal{P}_n^\times$, the exponential connection $\nabla^{(1)}$ gives a splitting of $T_{u_p}T\mathcal{P}_n^\times$ into the direct sum of the spaces of horizontal and vertical tangent vectors that one may identify with $T_p\mathcal{P}_n^\times \oplus T_p\mathcal{P}_n^\times$. With this splitting, the metric G at the point u_p is simply the direct sum $(g_F)_p \oplus (g_F)_p$, while the almost complex structure J is given, for $v_p, w_p \in T_p\mathcal{P}_n^\times$, by $J_{u_p}(v_p, w_p) := (-w_p, v_p)$. The resulting almost Hermitian structure will be referred to as the *almost splitting Hermitian structure* associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$.

Now, the link between the almost splitting Hermitian structure associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ and the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ goes as follows. As we show, there exists a covering map $\tau : T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times$ having the property that the "pull back" of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ is exactly the above almost splitting Hermitian structure on $T\mathcal{P}_n^\times$ (the latter being actually also a Kähler structure). Hence, we have a precise description of the statistical origin of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$.

A direct physical consequence –at least theoretically– is that a significant part

¹Information geometry is the branch of mathematics devoted to the study of statistics using differential geometrical tools (see [1]).

²A statistical manifold S is a manifold having the property that to every point $x \in S$ is associated (in an injective way) a probability p on a fixed measurable space Ω .

³In the definition of $\mathbb{P}(\mathbb{C}^n)^\times$ given above, we use homogeneous coordinates $[z_1, \dots, z_n] := \mathbb{C} \cdot z \subseteq \mathbb{C}^n$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n - \{0\}$.

of the quantum mechanical formalism (in finite dimension) is encoded in the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ via its associated almost splitting Hermitian structure. This physical consequence is, however, not discussed in this paper for which we refer the reader to [4].

This paper is organized as follows. In §2, we recall the definition of the Fisher metric g_F and the exponential connection $\nabla^{(1)}$ on the space of probabilities \mathcal{P}_n^\times , and we define the almost splitting Hermitian structure associated to $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$. In §3, we introduce a covering map $\tau : T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times$ and show that the “pull back” of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)^\times$ via τ is exactly the almost splitting Hermitian structure on $T\mathcal{P}_n^\times$ associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ (Proposition 3.3).

2 Information geometry and the statistical model \mathcal{P}_n^\times

Our statistical model will be the space \mathcal{P}_n^\times of non-vanishing probability distributions p on a discrete set $E_n := \{x_1, \dots, x_n\}$:

$$\mathcal{P}_n^\times := \left\{ p : E_n \rightarrow \mathbb{R} \mid p(x_i) > 0 \text{ for all } x_i \in E_n \text{ and } \sum_{i=1}^n p(x_i) = 1 \right\}. \quad (1)$$

This space is clearly a connected manifold of dimension $n - 1$. From a topological point of view, it is trivial since it can be realized as the interior of a simplex, but from a statistical point of view, it is naturally endowed with nontrivial geometric structures that we now want to describe (see [1]).

First of all, and for the rest of this paper, we will always use the so-called exponential representation for the tangent space :

$$T_p \mathcal{P}_n^\times \cong \{ u = (u_1, \dots, u_n) \in \mathbb{R}^n \mid u_1 p_1 + \dots + u_n p_n = 0 \}, \quad (2)$$

where $p \in \mathcal{P}_n^\times$, and where by definition, $p_i := p(x_i)$ for all $x_i \in E_n$.

If $u \in \mathbb{R}^n$ is a vector satisfying $u_1 p_1 + \dots + u_n p_n = 0$ for a given probability p , then we shall denote by $[u]_p$ the unique tangent vector of \mathcal{P}_n^\times at the point p determined by the exponential representation. One easily sees that if $p(t)$ is a smooth curve in \mathcal{P}_n^\times , then

$$\left. \frac{d}{dt} \right|_0 p(t) = [u]_{p(0)} \Leftrightarrow \left. \frac{d}{dt} \right|_0 p_i(t) = p_i(0) u_i \text{ for all } i = 1, \dots, n, \quad (3)$$

where $p_i(t) := (p(t))(x_i)$.

Equation (3) is actually one way to define the exponential representation.

The Fisher metric g_F is now defined, for $[u]_p, [v]_p \in T_p \mathcal{P}_n^\times$, by⁴

$$(g_F)_p([u]_p, [v]_p) := \frac{1}{4} \sum_{k=1}^n p_k u_k v_k = \frac{1}{4} E_p(uv), \quad (4)$$

⁴The Fisher metric is actually defined only up to a multiplicative constant, and thus we are free, for later convenience, to introduce the factor $1/4$.

where $E_p(\cdot)$ denotes the expectation with respect to the probability p and where uv denotes the vector in \mathbb{R}^n whose k -th component is $u_k v_k$. In most interesting statistical models, the Fisher metric is naturally and intrinsically defined, and is of central importance in information geometry (see [1]).

For a real parameter α , we also introduce the so-called α -connection $\nabla^{(\alpha)}$ on the space \mathcal{P}_n^\times . One way to define it is to use its associated covariant derivative along curves $D^{(\alpha)}/dt$: if $[V(t)]_{p(t)}$ is a vector field along a curve $p(t)$ in \mathcal{P}_n^\times such that $dp(t)/dt = [u(t)]_{p(t)}$, then, by definition,

$$\frac{D^{(\alpha)}}{dt}[V(t)]_{p(t)} := \left[\dot{V}(t) + (1 - \alpha)/2 \cdot u(t)V(t) - E_{p(t)}\left(\dot{V}(t) + (1 - \alpha)/2 \cdot u(t)V(t)\right) \cdot n \right]_{p(t)}, \quad (5)$$

where u and V are viewed as maps $\mathbb{R} \rightarrow \mathbb{R}^n$, $\dot{V}(t)$ is the usual derivative of $V(t)$ with respect to t and where $n := (1, \dots, 1) \in \mathbb{R}^n$.

It may be shown that $\nabla^{(0)}$ corresponds to the Levi-Civita connection associated to the Fisher metric g_F and also, if X, Y, Z are vector fields on \mathcal{P}_n^\times , that

$$X g_F(Y, Z) = g_F(\nabla_X^{(\alpha)} Y, Z) + g_F(Y, \nabla_X^{(-\alpha)} Z). \quad (6)$$

Because of (6), the triple $(g_F, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ is called a dualistic structure on \mathcal{P}_n^\times , and is also one of the major tools in information geometry (see [1]).

In the sequel, we will not use this dualistic structure, but we will restrict our attention to the 1-connection $\nabla^{(1)}$, also called exponential connection. This connection is probably, in view of (5), the simplest and the most natural connection among the family of α -connections since its expression reduces to

$$\frac{D^{(1)}}{dt}[V(t)]_{p(t)} = [\dot{V}(t) - E_{p(t)}(\dot{V}(t)) \cdot n]_{p(t)}. \quad (7)$$

We now want to describe the natural geometry of $T\mathcal{P}_n^\times$. Recall that if M is a manifold endowed with an affine connection ∇ , then the Dombrowski splitting Theorem holds (see [3]):

$$T(TM) \cong TM \oplus TM \oplus TM, \quad (8)$$

this splitting being viewed as an isomorphism of vector bundles over M , and the isomorphism, say Φ_M , being

$$T_{u_x} TM \ni A_{u_x} \xrightarrow{\Phi_M} (u_x, (\pi^{TM})_{*u_x} A_{u_x}, K A_{u_x}), \quad (9)$$

where $\pi^{TM} : TM \rightarrow M$ is the canonical projection and where $K : T(TM) \rightarrow TM$ is the canonical connector associated to the connection ∇ .

Applied to the couple $(\mathcal{P}_n^\times, \nabla^{(1)})$, the Dombrowski splitting Theorem yields

$$T(T\mathcal{P}_n^\times) \cong T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times. \quad (10)$$

It turns out that the inverse of the vector bundle isomorphism $\Phi_{\mathcal{P}_n^\times} : T(T\mathcal{P}_n^\times) \rightarrow T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times \oplus T\mathcal{P}_n^\times$ can be expressed very explicitly :

Lemma 2.1. *For $[u]_p, [v]_p, [w]_p \in T_p\mathcal{P}_n^\times$, we have :*

$$\Phi_{\mathcal{P}_n^\times}^{-1}([u]_p, [v]_p, [w]_p) = \frac{d}{dt} \Big|_0 \left[u + tw - E_{p(t)}(u + tw) \cdot n \right]_{p(t)}, \quad (11)$$

where $p(t)$ is a smooth curve in \mathcal{P}_n^\times satisfying $p(0) = p$ and $dp(t)/dt|_0 = [v]_p$.

Proof. It suffices to show that

$$\Phi_{\mathcal{P}_n^\times} \left(\frac{d}{dt} \Big|_0 \left[u + tw - E_{p(t)}(u + tw) \cdot n \right]_{p(t)} \right) = ([u]_p, [v]_p, [w]_p). \quad (12)$$

To this end, let us consider the curve γ in $T\mathcal{P}_n^\times$ which is defined by

$$\gamma(t) := \left[u + tw - E_{p(t)}(u + tw) \cdot n \right]_{p(t)}. \quad (13)$$

We have

$$\bullet \quad \gamma(0) = [u]_p, \quad (14)$$

$$\bullet \quad (\pi^{T\mathcal{P}_n^\times})_{*[u]_p} \frac{d}{dt} \Big|_0 \gamma(t) = \frac{d}{dt} \Big|_0 (\pi^{T\mathcal{P}_n^\times} \circ \gamma)(t) = \frac{d}{dt} \Big|_0 p(t) = [v]_p, \quad (15)$$

and thus, we immediately see that

$$\Phi_{\mathcal{P}_n^\times} \left(\frac{d}{dt} \Big|_0 \gamma(t) \right) = ([u]_p, [v]_p, *), \quad (16)$$

where “ $*$ ” has to be determined. But, according to (7), (9) and the fact that $K^{(1)}$ is a connector (here $K^{(1)}$ is the connector associated to the connection $\nabla^{(1)}$),

$$\begin{aligned} * &= K^{(1)} \frac{d}{dt} \Big|_0 \gamma(t) = \frac{D^{(1)}}{dt} \Big|_0 (\gamma(t)) = \left[\frac{d}{dt} \Big|_0 \left(u + tw - E_{p(t)}(u + tw) \cdot n \right) \right. \\ &\quad \left. - E_p \left(\frac{d}{dt} \Big|_0 \left(u + tw - E_{p(t)}(u + tw) \cdot n \right) \right) \cdot n \right]_p \\ &= \left[w - \frac{d}{dt} \Big|_0 E_{p(t)}(u + tw) \cdot n + \left(\frac{d}{dt} \Big|_0 E_{p(t)}(u + tw) \right) \cdot E_p(n) \cdot n \right]_p = [w]_p. \end{aligned} \quad (17)$$

Hence $\Phi_{\mathcal{P}_n^\times} (d\gamma(t)/dt|_0) = ([u]_p, [v]_p, [w]_p)$. The lemma follows. \square

In the sequel, we will identify a triple $([u]_p, [v]_p, [w]_p)$ with the corresponding element of $T_{[u]_p}(T\mathcal{P}_n^\times)$ via the isomorphism $\Phi_{\mathcal{P}_n^\times}$.

Having the decomposition (10), it is a simple matter to define on $T\mathcal{P}_n^\times$ an almost Hermitian structure. Indeed, we define a metric G , a 2-form Ω and an almost complex structure J by setting

$$\begin{aligned} G_{[u]_p} \left(([u]_p, [v]_p, [w]_p), ([u]_p, [\bar{v}]_p, [\bar{w}]_p) \right) &:= (g_F)_p([v]_p, [\bar{v}]_p) + (g_F)_p([w]_p, [\bar{w}]_p), \\ \Omega_{[u]_p} \left(([u]_p, [v]_p, [w]_p), ([u]_p, [\bar{v}]_p, [\bar{w}]_p) \right) &:= (g_F)_p([v]_p, [\bar{w}]_p) - (g_F)_p([w]_p, [\bar{v}]_p), \\ J_{[u]_p} \left(([u]_p, [v]_p, [w]_p) \right) &:= ([u]_p, -[w]_p, [v]_p), \end{aligned} \quad (18)$$

where $[u]_p, [v]_p, [w]_p, [\bar{v}]_p, [\bar{w}]_p \in T_p\mathcal{P}_n^\times$.

Clearly, $J^2 = -\text{Id}$ and $G(J\cdot, J\cdot) = G(\cdot, \cdot)$, which means that $(T\mathcal{P}_n^\times, G, J)$ is an almost Hermitian manifold, and one readily sees that G, J and Ω are compatible, i.e., that $\Omega = G(J\cdot, \cdot)$; the 2-form Ω is thus the fundamental 2-form of the almost Hermitian manifold $(T\mathcal{P}_n^\times, G, J)$.

The above geometric construction is a particular case of a more general construction which is due to Dombrowski (see [3]).

3 The statistical nature of $\mathbb{P}(\mathbb{C}^n)$

Recall that the projective space $\mathbb{P}(\mathbb{C}^n)$ is simply the quotient $(\mathbb{C}^n - \{0\})/\sim$, where the equivalence relation “ \sim ” is defined by

$$(z_1, \dots, z_n) \sim (w_1, \dots, w_n) \iff \exists \lambda \in \mathbb{C} - \{0\} : (z_1, \dots, z_n) = \lambda(w_1, \dots, w_n). \quad (19)$$

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n - \{0\}$, we shall denote by $[z] = [z_1, \dots, z_n]$ the corresponding element of $\mathbb{P}(\mathbb{C}^n)$. One may identify $[z]$ with the complex line $\mathbb{C} \cdot z$.

The manifold structure of $\mathbb{P}(\mathbb{C}^n)$ may be defined as follows. For a vector $u = (u_1, \dots, u_n) \in \mathbb{C}^n$ such that $|u|^2 = \langle u, u \rangle = \bar{u}_1 u_1 + \dots + \bar{u}_n u_n = 1$ (our convention for the Hermitian product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^n is that $\langle \cdot, \cdot \rangle$ is linear in the second argument), we define a chart (U_u, ϕ_u) of $\mathbb{P}(\mathbb{C}^n)$ by letting

$$\begin{cases} U_u := \{[z] \in \mathbb{P}(\mathbb{C}^n) \mid [u] \cap [z] = \{0\}\}, \\ \phi_u : U_u \rightarrow [u]^\perp \subseteq \mathbb{C}^n, [z] \mapsto \frac{1}{\langle u, z \rangle} \cdot z - u. \end{cases} \quad (20)$$

If u varies among all the unit vectors in \mathbb{C}^n , then the corresponding charts (U_u, ϕ_u) form an atlas for $\mathbb{P}(\mathbb{C}^n)$; the projective space is thus a real manifold of dimension $2(n-1)$, and, using the above charts, we have the identification

$$T_{[u]}\mathbb{P}(\mathbb{C}^n) \cong [u]^\perp = \{w \in \mathbb{C}^n \mid \langle u, w \rangle = 0\}. \quad (21)$$

The Fubini-Study metric g_{FS} and the Fubini-Study symplectic form ω_{FS} are now defined at the point $[u]$ in $\mathbb{P}(\mathbb{C}^n)$ via the formulas :

$$\left((\phi_u^{-1})^* g_{FS} \right)_0 (\xi_1, \xi_2) := \text{Re} \langle \xi_1, \xi_2 \rangle, \quad \left((\phi_u^{-1})^* \omega_{FS} \right)_0 (\xi_1, \xi_2) := \text{Im} \langle \xi_1, \xi_2 \rangle, \quad (22)$$

where $\xi_1, \xi_2 \in [u]^\perp \cong T_{[u]}\mathbb{P}(\mathbb{C}^n)$.

One may show that g_{FS} and ω_{FS} are globally well defined on $\mathbb{P}(\mathbb{C}^n)$.

We now want to relate the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ with the almost splitting Hermitian structure associated to the triple $(\mathcal{P}_n^\times, g_F, \nabla^{(1)})$ discussed in §2. To this end, we set

$$\mathbb{P}(\mathbb{C}^n)^\times := \{ [z_1, \dots, z_n] \in \mathbb{P}(\mathbb{C}^n) \mid z_i \neq 0 \text{ for all } i = 1, \dots, n \} \quad (23)$$

and introduce the following smooth map

$$\tau : T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times, \quad [u]_p \mapsto [\sqrt{p_1} e^{iu_1/2}, \dots, \sqrt{p_n} e^{iu_n/2}]. \quad (24)$$

The geometrical nature of the map τ is given by the following lemma.

Lemma 3.1. *The map $\tau : T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times$ is a universal covering map whose deck transformation group is a copy of \mathbb{Z}^{n-1} .*

Proof. Let \mathbb{T}^{n-1} denotes the $(n-1)$ -dimensional torus, and let us consider the following diagram :

$$\begin{array}{ccc} T\mathcal{P}_n^\times & \xrightarrow{\tau} & \mathbb{P}(\mathbb{C}^n)^\times \\ j_1 \downarrow & & \downarrow j_2 \\ \mathcal{P}_n^\times \times \mathbb{R}^{n-1} & \xrightarrow{\bar{\tau}} & \mathcal{P}_n^\times \times \mathbb{T}^{n-1} \end{array} \quad (25)$$

where

- $j_1([u]_p) := (p, (u_1 - u_n, \dots, u_{n-1} - u_n))$,
- $j_2([\sqrt{p_1} e^{iu_1/2}, \dots, \sqrt{p_n} e^{iu_n/2}]) := (p, (e^{i(u_1 - u_n)/2}, \dots, e^{i(u_{n-1} - u_n)/2}))$,
- $\bar{\tau}(p, u) := (p, (e^{iu_1/2}, \dots, e^{iu_{n-1}/2}))$.

Clearly, j_1 and j_2 are diffeomorphisms, and one easily sees that $\bar{\tau}$ is nothing but the quotient map associated to the (free and proper) action of the group \mathbb{Z}^{n-1} on $\mathcal{P}_n^\times \times \mathbb{R}^{n-1}$ given by $(k_1, \dots, k_{n-1}) \cdot (p, u) := (p, (u_1 + 4k_1\pi, \dots, u_{n-1} + 4k_{n-1}\pi))$. As the diagram is manifestly commutative, the lemma follows. \square

Lemma 3.2. *For $([u]_p, [v]_p, [w]_p) \in T_{[u]_p}T\mathcal{P}_n^\times$, we have*

$$\begin{aligned} & (\phi_z \circ \tau)_{*_{[u]_p}}([u]_p, [v]_p, [w]_p) \\ &= \left(1/2 \sqrt{p_1} e^{iu_1/2} (v_1 + iw_1), \dots, 1/2 \sqrt{p_1} e^{iu_n/2} (v_n + iw_n) \right), \end{aligned} \quad (26)$$

where $z := (\sqrt{p_1} e^{iu_1/2}, \dots, \sqrt{p_n} e^{iu_n/2}) \in \mathbb{C}^n$ and where $\phi_z : U_z \subseteq \mathbb{P}(\mathbb{C}^n) \rightarrow [z]^\perp \subseteq \mathbb{C}^n$ is the chart on $\mathbb{P}(\mathbb{C}^n)$ introduced in (20).

Proof. Let us fix $[u]_p, [v]_p, [w]_p \in T\mathcal{P}_n^\times$ and set

$$z(t) := (\sqrt{p_1(t)} e^{i(u_1+tw_1)/2}, \dots, \sqrt{p_n(t)} e^{i(u_n+tw_n)/2}) \in \mathbb{C}^n - \{0\}, \quad (27)$$

where $p(t)$ is a smooth curve in \mathcal{P}_n^\times satisfying $dp(t)/dt|_0 = [v]_p$. Observe that $[z] = [z(0)] = \tau([u]_p)$, $\langle z(t), z(t) \rangle^2 = 1$ and that $\langle z(t), \dot{z}(t) \rangle = 0$ since $z(t)$ is normalized. Using the identification (10), Lemma 2.1 as well as the formula $[\lambda \cdot z] = [z]$ ($\lambda \in \mathbb{C} - \{0\}$) which holds on $\mathbb{P}(\mathbb{C}^n)$, we see that

$$\begin{aligned} & (\phi_{z(0)} \circ \tau)_{*[u]_p}([u]_p, [v]_p, [w]_p) = \frac{d}{dt} \Big|_0 (\phi_{z(0)} \circ \tau) \left([u+tw - E_{p(t)}(u+tw) \cdot n]_{p(t)} \right) \\ &= \frac{d}{dt} \Big|_0 \phi_{z(0)} \left([\sqrt{p_1(t)} e^{i(u_1+tw_1)/2} e^{-iE_{p(t)}(u+tw)/2}, \dots \right. \\ & \quad \left. \dots, \sqrt{p_n(t)} e^{i(u_n+tw_n)/2} e^{-iE_{p(t)}(u+tw)/2}] \right) \\ &= \frac{d}{dt} \Big|_0 \phi_{z(0)} \left([\sqrt{p_1(t)} e^{i(u_1+tw_1)/2}, \dots, \sqrt{p_n(t)} e^{i(u_n+tw_n)/2}] \right) \\ &= \frac{d}{dt} \Big|_0 \phi_{z(0)}([z(t)]) = \frac{d}{dt} \Big|_0 \left(\frac{1}{\langle z(0), z(t) \rangle} \cdot z(t) - z(0) \right) \\ &= \frac{-\langle z(0), \dot{z}(0) \rangle}{\langle z(0), z(0) \rangle^2} z(0) + \frac{1}{\langle z(0), z(0) \rangle} \dot{z}(0) = -\langle z(0), \dot{z}(0) \rangle z(0) + \dot{z}(0) = \dot{z}(0). \quad (28) \end{aligned}$$

The lemma is now a direct consequence of

$$\begin{aligned} (\dot{z}(0))_j &= \frac{d}{dt} \Big|_0 \sqrt{p_j(t)} e^{i(u_j+tw_j)/2} = \frac{1}{2\sqrt{p_j}} p_j v_j e^{iu_j/2} + \sqrt{p_j} \frac{i}{2} w_j e^{iu_j/2} \\ &= 1/2 \sqrt{p_j} e^{iu_j/2} (v_j + iw_j) \end{aligned} \quad (29)$$

(of course, in the above computations we use extensively the exponential representation (3)). \square

For the next proposition (which is the main observation of this paper), recall that $T\mathcal{P}_n^\times$ is endowed with its associated splitting Hermitian structure (G, J, Ω) introduced at the end of §2 and that $\mathbb{P}(\mathbb{C}^n)$ possesses its natural Kähler structure $(g_{FS}, J_{FS}, \omega_{FS})$.

Proposition 3.3. *The covering map $\tau : T\mathcal{P}_n^\times \rightarrow \mathbb{P}(\mathbb{C}^n)^\times$ defined in (24) has the following properties :*

$$\tau^* g_{FS} = G, \quad \tau^* \omega_{FS} = \Omega, \quad \tau_* J = J_{FS} \tau_*. \quad (30)$$

Proof. Let us fix $[u]_p, [v]_p, [w]_p, [\bar{v}]_p, [\bar{w}]_p \in T_p\mathcal{P}_n^\times$ and define the normalized vector

$$z := (\sqrt{p_1} e^{iu_1/2}, \dots, \sqrt{p_n} e^{iu_n/2}) \in \mathbb{C}^n. \quad (31)$$

According to Lemma 3.2, we have :

$$\begin{aligned}
& \left\langle (\phi_z \circ \tau)_{*[u]_p}([u]_p, [v]_p, [w]_p), (\phi_z \circ \tau)_{*[u]_p}([u]_p, [\bar{v}]_p, [\bar{w}]_p) \right\rangle \\
&= \left\langle \left(1/2 \sqrt{p_1} e^{iu_1/2} (v_1 + iw_1), \dots, 1/2 \sqrt{p_1} e^{iu_n/2} (v_n + iw_n) \right) \right. \\
&\quad \left. , \left(1/2 \sqrt{p_1} e^{iu_1/2} (\bar{v}_1 + i\bar{w}_1), \dots, 1/2 \sqrt{p_1} e^{iu_n/2} (\bar{v}_n + i\bar{w}_n) \right) \right\rangle \\
&= \sum_{j=1}^n \frac{1}{4} p_j (v_j - iw_j)(\bar{v}_j + i\bar{w}_j) = \sum_{j=1}^n \frac{1}{4} p_j (v_j \bar{v}_j + w_j \bar{w}_j + i(v_j \bar{w}_j - \bar{v}_j w_j)) \\
&= \frac{1}{4} \sum_{j=1}^n p_j (v_j \bar{v}_j + w_j \bar{w}_j) + \frac{i}{4} \sum_{j=1}^n p_j (v_j \bar{w}_j - \bar{v}_j w_j) \\
&= G_{[u]_p}([u]_p, [v]_p, [w]_p), ([u]_p, [\bar{v}]_p, [\bar{w}]_p) \\
&\quad + i \Omega_{[u]_p}([u]_p, [v]_p, [w]_p), ([u]_p, [\bar{v}]_p, [\bar{w}]_p) . \tag{32}
\end{aligned}$$

Comparing (32) with the definition of g_{FS} and ω_{FS} given in (22) gives the first two relations in (30), and these two relations, together with the fact that both triples (G, J, Ω) and $(g_{FS}, J_{FS}, \omega_{FS})$ are compatible, imply the last relation in (30). The proposition follows. \square

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